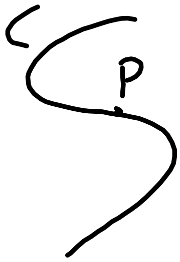
§ Extensions of absolute values: examples

Def $K, |\cdot|_K \hookrightarrow L, |\cdot|_L$

$|\cdot|_L$ extends $|\cdot|_K$ if $|\cdot|_L\big|_K \sim |\cdot|_K$. (same for valuations)

Example 1 Maps of algebraic curves $/k$ $[=\mathbb{C}]$



non-singular curve, e.g. $f(x,y)=0$

$\mathcal{O} = k[C] = \frac{k[x,y]}{f}$ ring of regular fncs

$K = k(C) = \text{f.f.}(\mathcal{O})$ field of rational fncs.

points $P \in C$ $\xleftrightarrow{1:1}$ maximal ideal $m_P = (x-a, y-b)$
$\quad\; \shortparallel \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \cap$
$\;(a,b) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; k[C]$

$\longleftrightarrow$ valuations on $k(C)$, trivial on k.

expand $f(x,y)$ about $P=(a,b)$

$$f(x,y) = \underbrace{f(P)}_{0} + f_x'(P)(x-a) + f_y'(P)(y-b) +$$

$$+ \text{higher order terms} \in m_P^2$$

P non-singular $\Leftrightarrow$ $f_x'(P), f_y'(P)$ not both 0

$\Leftrightarrow$ relation between $x-a$, $y-b$ in m_P/m_P^2

$\Leftrightarrow$ $\dim m_P/m_P^2 = 1$ $\Leftrightarrow$

$\Leftrightarrow \mathcal{O}_{C,P} = \left\{ \frac{g}{h} \in k(C) \,\middle|\, h(P) \neq 0 \right\}$

is a DVR (local integral domain, dim. 1, m/m^2 1-dim)

$\Rightarrow$ valuation ord_P (or v_P) on $k(C)$

Completions:

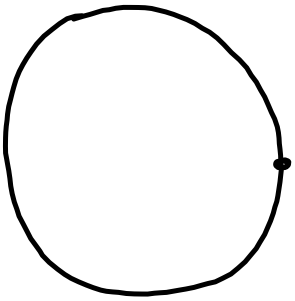
$$\hat{\mathcal{O}} = \varprojlim_n \mathcal{O}/m_P^n \cong k[[\pi]]$$

π any uniformiser at P (generator of m_P/m_P^2)

same proof as classification of local equal char. fields.

$$\hat{K} \cong k((\pi)).$$

Ex $C: x^2+y^2=1$



at $P=(1,0)$

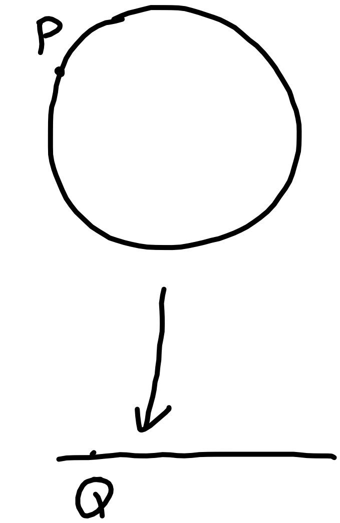
$$x^2+y^2-1 = \underbrace{2}_{f_x'(P)}(x-1) + \underbrace{0}_{f_y'(P)}\cdot y + \underbrace{y^2+(x-1)^2}_{\in m_P^2}$$

$\Rightarrow m_P/m_P^2 = \langle y \rangle$ $\quad$ y uniformizer at P

$v_P(y)=1$, $\quad v_P(x-1) = v_P((x-1)\overbrace{(x+1)}^{\text{unit}})$

$= v_P(y^2) = 2.$

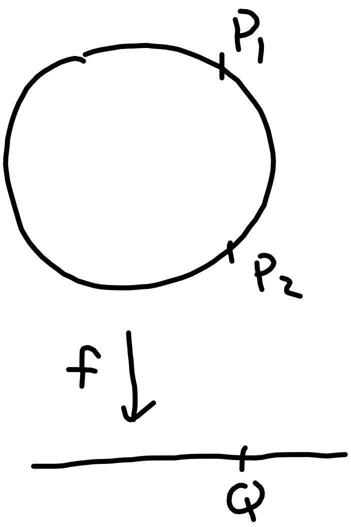
Maps between curves:



$$\begin{tikzcd} C: x^2+y^2=1 \arrow[d, "f=x"] \\ \mathbb{A}^1 \ \text{x-line} \end{tikzcd} \qquad \begin{tikzcd} \dfrac{k[x,y]}{x^2+y^2-1} \\ k[x] \arrow[u, hook, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} k(x,\sqrt{1-x^2}) \\ k(x) \arrow[u, hook, "\deg\ 2."'] \end{tikzcd}$$

$f(P)=Q \iff \mathrm{ord}_Q \sim f^* \circ \mathrm{ord}_P$

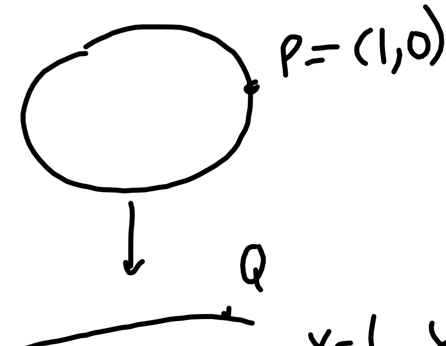
$f^* \circ \mathrm{ord}_P = e \cdot \mathrm{ord}_Q$ for some $e \geq 1$, called the <u>ramification index</u> e_P.



$f^*(x-a) = x-a$ again a uniformiser at P_1, P_2

$\Rightarrow e_{P_1} = e_{P_2} = 1.$

$x-a$ uniformiser at Q



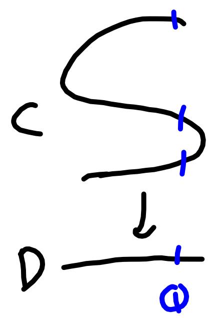
$f^*(x-1) = x-1$ has valuation 2

$\Rightarrow e_P = 2.$

$x-1$ uniformiser

Prop $C \xrightarrow{f} D$ non-constant map of non-singular projective algebraic curves $/k = \overline{k}$, of degree $d = [k(C) : f^*k(D)]$

For every $Q \in D$,



$$\sum_{P \text{ s.t. } f(P)=Q} e_P = d.$$

Taking completions of $k(D) \xhookrightarrow{f^*} k(C)$ at P, Q,

$$\widehat{k(D)} \xhookrightarrow{e} \widehat{k(C)}$$

If char $k = 0$, this is simply

$$k((\pi)) \hookrightarrow k((\pi^{1/e})) \qquad \pi \text{ uniformizer at } Q.$$

<u>Rmk</u> If $k \neq \bar{k}$, then the residue field

$$k_p = \frac{\mathcal{O}_{C,p}}{m_p}$$ may be a (finite) extension of k.

[recall: valuations of $k(x)$ correspond to irr. polys $F \subseteq k[x]$, res. field

$k[x]/F \leftarrow$ fin. ext. of k of degree $\deg F$].

and

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ P & \longmapsto & Q \\ k_P & \hookleftarrow & k_Q \end{array}$$

finite extension of some degree f_P, called the <u>residue degree</u>

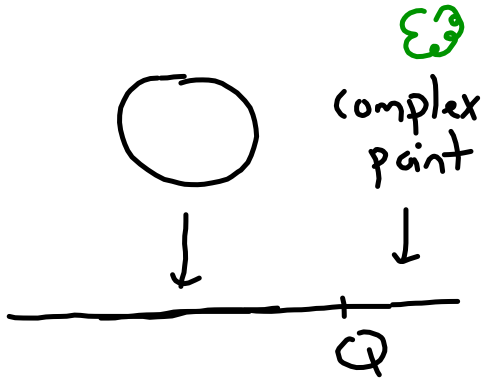
Prop. above becomes

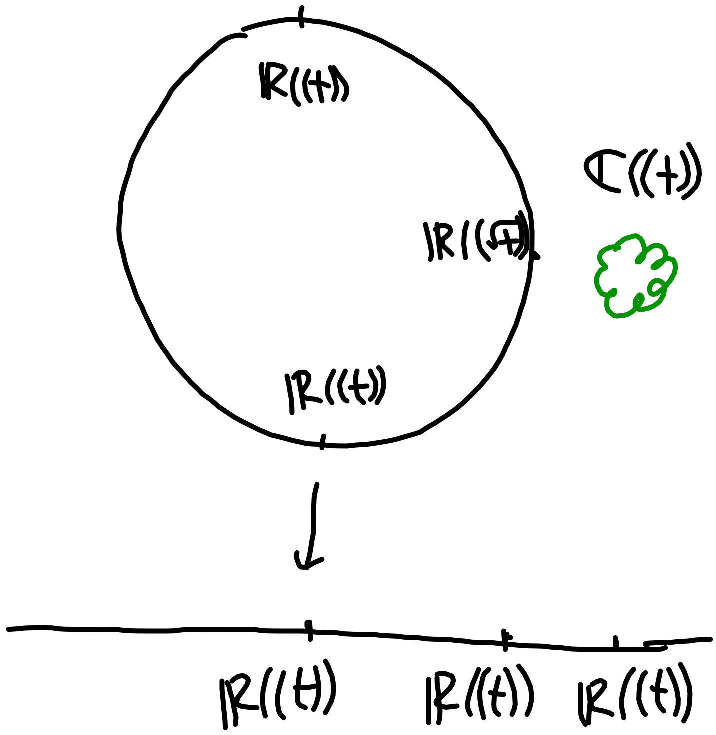
$$\sum_{f(P)=Q} e_P f_P = d \quad , \quad k_Q((\pi)) \xhookrightarrow{e_P f_P} k_P((\pi^{1/e}))$$

completions

(char $k = 0$)

Ex $k = \mathbb{R}$





Example 2. Number fields

$$\begin{array}{ccc} L & \mathcal{O}_L & \mathfrak{p} \\ \big\uparrow \text{deg. } d & \big\uparrow & \\ K & \mathcal{O}_K & \mathfrak{q} \\ \text{number fields} & \text{rings of integers} & \end{array}$$

valuations $\overset{1:1}{\longleftrightarrow}$ prime ideals

$|\cdot|_{\mathfrak{p}}$ on L extends $|\cdot|_{\mathfrak{q}}$ on K

$\Longleftrightarrow$

$\mathfrak{p} \cap \mathcal{O}_K = \mathfrak{q}$

$\Longleftrightarrow$

$\mathfrak{p} \mid \mathfrak{q}$

In $\mathcal{O}_L$ factor $\mathfrak{q} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n}$ prime ideals in $\mathcal{O}_L$

$e_i = e_{\mathfrak{p}_i}$ is <u>ramification index</u>, and

$$f_i = f_{\mathfrak{P}_i} = \left[\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{q} \right]$$ <u>residue degree</u>.

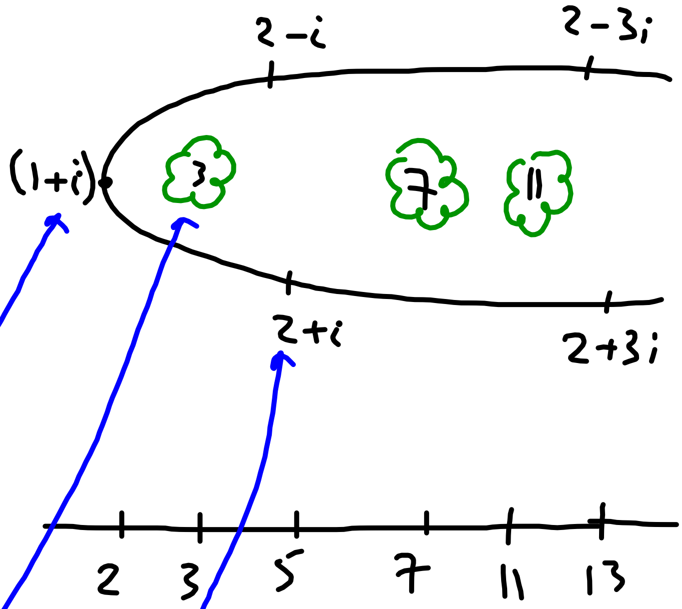
<u>Thm</u> For every prime ideal $\mathfrak{q} \subseteq \mathcal{O}_K$,

$$\sum_{\mathfrak{P}_i \mid \mathfrak{q}} e_{\mathfrak{P}_i} f_{\mathfrak{P}_i} = d$$

Completions: $K_{\mathfrak{q}} \xhookrightarrow{e_{\mathfrak{P}_i} f_{\mathfrak{P}_i}} L_{\mathfrak{P}_i}$

<u>Ex</u>

$$\begin{array}{cc} \mathbb{Q}(i) & \mathbb{Z}[i] \\ \uparrow & \uparrow \\ \mathbb{Q} & \mathbb{Z} \end{array}$$



$e = 2$ (prime ramifies)

$f = 2$ (prime is inert)

2 primes (prime is split).

§ Extensions of complete fields

Thm K, $|\cdot|_K$ complete, L/K fin. ext. of degree d.

(i) $\exists!$ abs. value $|\cdot|_L$ on L extending $|\cdot|_K$.

(ii) L is complete w.r.to $|\cdot|_L$.

(iii) $|x|_L = |N_{L/K}(x)|_K^{1/d}$

(iv) If K non-Archimedean, then
$\mathcal{O}_L$ = integral closure of $\mathcal{O}_K$ in L
[$:= \{x \in L \mid x$ satisfies a monic poly eqn with $\mathcal{O}_K$ - coefficients$\}$.]

<u>Proof</u> Not hard.

§ <u>Consequences</u>

$K, |\cdot|$ non-Archimedean, $\mathcal{O}$.

<u>Cor</u> $|\cdot|$ extends uniquely to $\overline{K}$

$$\alpha \in \overline{K} \Rightarrow |\alpha| = |N_{L/K}(\alpha)|^{1/[L:K]}$$

for any finite field ext. L of $K(\alpha)$,

e.g. $L = K(\alpha)$

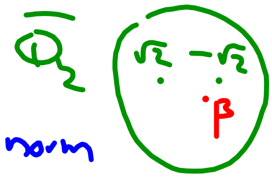
<u>Rmk</u> $\overline{K}$ in general not complete (e.g. $\overline{\mathbb{Q}_p}$ is not)

but its completion $\widehat{\overline{K}}$ is complete <u>and</u> algebraically closed. (denoted $\mathbb{C}_p$ for $K = \mathbb{Q}_p$).

<u>Cor</u> $\alpha, \alpha' \in \overline{K}$ Galois conjugate over $K \Rightarrow |\alpha| = |\alpha'|$.

<u>Pf</u> Gal. conjugate $\Rightarrow$ roots of the same monic irr. poly. $f(x) \in K[x]$

$\Rightarrow$ same norm $f(0)$.



norm

__Ex__ $K = \mathbb{Q}_2$

$|2| = \frac{1}{2}$

$\sqrt{2}, -\sqrt{2} \in \overline{\mathbb{Q}_2}$ two roots of $x^2 - 2 = 0$

Here $|\sqrt{2}| = |-\sqrt{2}| = \frac{1}{\sqrt{2}}$

__Lemma__ (Krasner's Lemma) $f(x) \in K[x]$ monic irreducible, say $f(x) = \prod_{i=1}^{d} (x - \alpha_i)$, $\alpha_i \in \overline{K}$.

Suppose $\beta \in \overline{K}$ s.t. $|\beta - \alpha_1| < |\beta - \alpha_i|$ $i = 2, \ldots, d$

Then $\alpha_1 \in K(\beta)$.

<u>Proof</u> Assumption $\Rightarrow$ $\alpha_1 \neq \alpha_2, \dots, \alpha_d$, i.e. f is separable.

Consider

$$\begin{array}{rl} & L' := L(\alpha_1, \dots, \alpha_d) \\ \text{Galois}\left(\begin{array}{c} \\ \\ \end{array}\right. & \quad | \\ & L := K(\beta) \\ & \quad | \\ & \quad K \end{array}$$

For $\sigma \in \mathrm{Gal}(L/K)$

$$|\beta - \sigma(\alpha_1)| = |\sigma(\beta - \alpha_1)| = |\beta - \alpha_1| < |\beta - \alpha_i| \quad i = 2, \dots, d$$

$\Rightarrow \sigma(\alpha_1) \neq \alpha_2, \dots, \alpha_d$

$\Rightarrow \sigma(\alpha_1) = \alpha_1 \ \forall \sigma \Rightarrow \alpha \in K(\beta)$

$\blacksquare$.

Cor (Nearby polynomials define the same field)

$f(x) \in K[x]$ monic, separable, irreducible, $\deg f = d$.

There is an open nbd Ω of f in

$$K^d = \{ x^d + a_{d-1} x^{d-1} + \dots + a_0 \mid a_i \in K \}$$

s.t. for every $g \in \Omega$, g is irr., separable, and $\frac{K[x]}{f} \cong \frac{K[x]}{g}$.

Pf Suppose not, choose $f^{(i)} \longrightarrow f$ in K^d

s.t. $f^{(i)}$ define different fields from f.

$\alpha_j^{(i)}$ roots of $f^{(i)}$; α_j roots of f

$j = 1, \ldots, d$.

Recall: $\Delta_f = \prod_{i<j} (\alpha_i - \alpha_j)^2$ discriminant

$\mathrm{Res}(f^{(i)}, f) = \prod_{j,k} (\alpha_j - \alpha_k^{(i)})$ resultant

polynomials in coefficients of f and $f^{(i)}$

$f^{(i)} \to f \implies \Delta_{f_i} \to \Delta_f \neq 0$ (f separable)

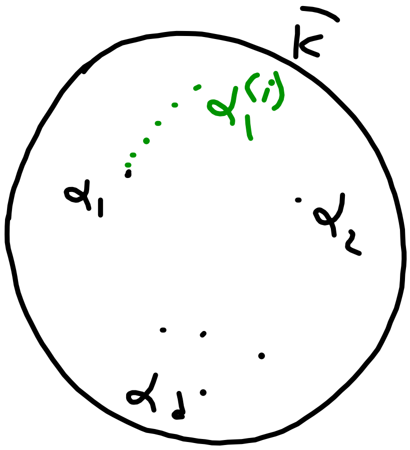
$\implies \Delta_{f_i} \neq 0$ for i large,

i.e. $f^{(i)}$ separable.

$\mathrm{Res}(f^{(i)}, f) \to \mathrm{Res}(f, f) = 0 \implies$

renumbering the roots can find a subsequence

with $|\alpha_1^{(i)} - \alpha_1| \to 0$



in particular

$$|\alpha_1^{(i)} - \alpha_1| < |\alpha_1^{(i)} - \alpha_j| \qquad j = 2, \dots, d$$

for i large enough.

Krasner's Lemma $\Rightarrow$ $\underbrace{\alpha_1}_{\text{degree } d \text{ over } K} \in \underbrace{K(\alpha_1^{(i)})}_{\text{degree} \le d \text{ over } K}$

$\Rightarrow$ $f^{(i)}$ irreducible & $K(\alpha_1) = K(\alpha_1^{(i)})$.

contradiction $\blacksquare$

Note Can be made completely explicit.

Ex $\mathbb{Q}_3(i)$ quad. ext. of $\mathbb{Q}_3$, i root of $x^2+1=0$

Any x^2+ax+b $\quad a \equiv 0 \mod 3$, $b \equiv 1 \mod 3$

open set of polys

define the same field $\mathbb{Q}_3(\sqrt{a^2-4b})$

$\equiv -4 \mod 3$

$\equiv -1 \mod 3$; $-1 \times$ square unit

$= \mathbb{Q}_3(i)$.

<u>Cor</u> $\mathbb{Q}_p$ has only finitely many extensions of a given degree d (up to $\cong$).

<u>Proof</u> Not hard; essentially compactness of $\mathbb{Z}_p^d$)

The same is true for any local field for separable extensions of degree d.

<u>Exc</u> Prove that $\mathbb{F}_2((t))$ has infinitely many inseparable extensions of degree 2.